BOREL FIXED INITIAL IDEALS OF PRIME IDEALS IN DIMENSION TWO

AMELIA TAYLOR

ABSTRACT. We prove that if the initial ideal of a prime ideal is Borel-fixed and the dimension of the quotient ring is less than or equal to two, then given any non-minimal associated prime ideal of the initial ideal it contains another associated prime ideal of dimension one larger.

Let $R = k[x_1, x_2, \ldots, x_r]$ be a polynomial ring over a field. We will say that an ideal $I \subseteq R$ has the *saturated chain property* if given any non-minimal associated prime ideal Q of I there exists an associated prime ideal $P \subseteq Q$ such that $\dim(R/P) = \dim(R/Q)+1$. Therefore given Q an associated prime ideal of I, there exists a saturated chain of associated prime ideals $P_1 \subset P_2 \subset \cdots P_n = Q$ such that P_1 is minimal and $\dim(R/P_i) = \dim(R/P_{i+1}) + 1$ for $1 \leq i \leq n-1$.

In 1999 Hosten and Thomas [HT] proved that the initial ideal of a toric ideal has this saturated chain property. This type of connectivity does not exist in general for initial ideals of prime ideals as the following counter-example, provided by Hosten and S. Popescu, illustrates.

Counter Example 1. Take the toric ideal $\langle xz-a^2, yz-b^2, tz-c^2\rangle$ and substitute $z-t$ for t, making the ideal $\langle xz-a^2, yz-b^2, z^2-tz-c^2\rangle$ which is still a prime ideal. Using the reverse lexicographic order and the variable order $x > y > z > t > a > b > c$ the initial ideal of this prime ideal is $J = \langle z^2, yz, xz, zb^2, za^2, ytb^2, xtb^2, xta^2\rangle$. The primary decomposition of J is

$$J = \langle x, y, z\rangle \cap \langle y, t, z\rangle \cap \langle t, z, a^2\rangle \cap \langle z, a^2, b^2\rangle \cap \langle x, y, z^2, a^2, b^2\rangle.$$

Hence $Ass(R/J) = \{\langle x, y, z\rangle, \langle y, t, z\rangle, \langle t, a, z\rangle, \langle z, a, b\rangle, \langle x, y, z, a, b\rangle\}$.

Hosten and others have since constructed families of prime ideals such that the initial ideal of a prime ideal in the family does not have the saturated chain property. However, it was conjectured that lexicographic generic initial ideals of homogeneous prime ideals have the saturated chain property. We prove that if P is a homogeneous prime ideal of $R = k[x_1, x_2, \ldots, x_r]$, $\dim(R/P) = 2$ and the initial ideal of P is Borel-fixed, then the initial ideal of P has the saturated chain property. There are many prime ideals with Borel-fixed initial ideal since for any prime ideal P the generic initial ideal of P is Borel-fixed. While we restrict the dimension, we do not make any assumptions on the monomial order used, nor do we require that the initial ideal be generic, only Borel-fixed which is weaker.

We collect some key definitions and properties and then give the main theorem. A *monomial order* $\geq$ on a polynomial ring $R = k[x_1, x_2, \ldots, x_r]$ over a field k is a total order on the monomials in R such that $m \geq 1$ for each monomial m in R and if m_1, m_2, n are monomials in R with $m_1 \geq m_2$ then $nm_1 \geq nm_2$. A

The research of the author was partially supported by the National Security Agency.

monomial order on a polynomial ring in several variables generalizes the notion of degree for a polynomial ring in one variable. The *initial term* of an element $f \in R$, denoted $in(f)$, is the largest term of f with respect to a fixed monomial order. Given an ideal I of R, the *initial ideal* is defined to be $\langle \{in(f) : f \in I\} \rangle$, and is denoted $in(I)$. It should be noted that different monomial orders may yield different initial ideals, so whenever an initial ideal is referred to, it is assumed a monomial order has been fixed. A *Gröbner basis* is a subset $\{g_1, \dots, g_n\}$ of I such that $in(I) = \langle in(g_1), \dots, in(g_n) \rangle$.

Let $\mathcal{G} = Gl(r, k)$ be the $r \times r$ invertible matrices over the field k. Let $g = (g_{ij})$ be a matrix in $\mathcal{G}$. The *Borel subgroup* $B = \{(g_{ij}) \in \mathcal{G} \mid g_{ij} = 0 \text{ for } j > i\}$ is the subgroup of $\mathcal{G}$ of lower triangular matrices. We get an automorphism of R from g by describing how g acts on the variables. Define $g(x_i) = \sum_{j=1}^{r} g_{ij} x_j$. Notice that if m is a monomial $x_1^{j_1} \cdots x_r^{j_r}$ then $g(m) = \prod_l (\sum_k g_{lk} x_k)^{j_l}$. Define $g(I) = \{g(f) \mid f \in I\}$ to be the action of g on an ideal I in R. An ideal I is said to be *Borel-fixed* if $g(I) = I$ for every $g \in B$. Green [Gre] gives a more convenient way, for our purposes, of thinking about Borel-fixed ideals through the following definition and proposition.

Definition 2. [Gre, Definition 1.24] An elementary move e_k for $1 \leq k \leq n-1$ is defined by $e_k(x^J) = x^{\hat{J}}$ where $\hat{J} = (j_1, \dots, j_{k-1}, j_k + 1, j_{k+1} - 1, j_{k+2}, \dots, j_r)$ and where we adopt the convention that $x^J = 0$ if some $j_m < 0$.

Proposition 3. [Gre, Proposition 1.25] *Let I be a monomial ideal. The following are equivalent.*
(1) If $x^J \in I$, then for every elementary move $e_k(x^J) \in I$;
(2) $g(I) = I$ for every g belonging to the Borel subgroup;
(3) $in(g(I)) = I$ for every g in some open neighborhood of the identity in B.

The main property of Borel-fixed ideals that we want to get from this definition is that if $x_j m \in I$, where m is any monomial in R, then $x_i m \in I$ for every $i < j$. Another key property of Borel-fixed ideals involves the structure of the associated primes of R/I.

Corollary 4. [E, Corollary 15.25] *If I is a Borel-fixed ideal in R and P is an associated prime of R/I, then $P = (x_1, \dots, x_j)$ for some j. If $Q = (x_1, \dots, x_t)$ is a maximal associated prime then $x_{t+1}, \dots, x_r$ (in any order) is a maximal (R/I)-regular sequence in $(x_1, \dots, x_r)$.*

Let I be the initial ideal of a prime ideal $P \subseteq R = k[x_1, x_2, \dots, x_r]$. Assume $<$ is the reverse lexicographic order. A theorem of Bayer and Stillman [BS, Theorem 2.4], gives that the image of x_r, in R/I, is a non-zero divisor in R/I if and only if the image of x_r, in R/I, is a non-zero divisor in R/P. Using this and the fact that P is a prime ideal, x_r is a non-zero divisor in R/I and hence cannot be in any associated prime of R/I. Hence, if $codim(I) = r - 2$, that is if $\dim(R/P) = 2$, and I is a Borel-fixed ideal, then $Ass(R/I) = \{\langle x_1, \dots, x_{r-2} \rangle\}$ or $Ass(R/I) = \{\langle x_1, \dots, x_{r-2} \rangle, \langle x_1, \dots, x_{r-1} \rangle\}$. Since the initial ideal of a prime ideal is equidimensional [KS], a similar assessment of the possible choices for the set of associated prime ideals of such an initial ideal establishes that for any monomial order $\dim(R/P) = 2$ is the first interesting dimension for this question.

Theorem 5. *Let $R = k[x_1, x_2, \ldots, x_r]$ and P be a homogeneous prime ideal in R such that $\dim(R/P) = 2$. Fix a monomial order. Assume the initial ideal $in(P)$ is a Borel-fixed ideal. Then $in(P)$ has the saturated chain property.*

Proof. The ring R is regular, so $\dim(R/P) = 2$ implies $\operatorname{codim}(P) = r - 2$. Let $I = in(P)$, then the codimension of I is also $r-2$. The prime ideal $(x_1, \ldots, x_{r-2})$ is minimal over I, since I is equidimensional [KS], Borel-fixed and codimension $r-2$. Therefore $(x_1, \ldots, x_{r-2})$ is in $Ass(R/I)$. We assume $(x_1, \ldots, x_r) \in Ass(R/I)$, as otherwise, by Corollary 4, there is nothing to prove. We prove that $(x_1, \ldots, x_{r-1}) \in Ass(R/I)$.

Since $(x_1, \ldots, x_r) \in Ass(R/I)$ there exists a monomial $z \in R \setminus I$ such that $\operatorname{ann}_{(R/I)}(\overline{z}) = (\overline{x_1}, \ldots, \overline{x_r})$, where $\overline{x}$ denotes the image of x in R/I. All such z are in the socle of R/I which is a finite dimensional vector space and hence there is a monomial of maximal total degree such that its annihilator, in R/I, is exactly the maximal ideal $(\overline{x_1}, \ldots, \overline{x_r})$. Choose z to be a monomial of maximal total degree in the socle of R/I. Since $\operatorname{ann}_{R/I}(\overline{z}) = (\overline{x_1}, \ldots, \overline{x_r})$ we know $z(x_1, \ldots, x_r) \subseteq I$. Hence there exist $f_1, \ldots, f_r \in P$ homogeneous, such that $f_i = x_i z + h_i$ and $x_i z > in(h_i)$ for $1 \leq i \leq r$. We may assume $in(h_i) \notin I$, since if $in(h_i) \in I$, then there exists a homogeneous polynomial G in P such that $G = in(h_i) + g$ and $in(G) = in(h_i)$. The difference $f_i - G$ is in P and $in(f_i - G) = x_i z$. Repeat this process until a polynomial is obtained where all terms smaller than $x_i z$ are not in I replace f_i with this new polynomial. During this process the leading term of the polynomial did not change. Also, for $i = r-1, r$, f_i must have a second term, since if there is not a second term then $x_i z \in P$ implies either $x_i \in P$ or $z \in P$. If $z \in P$ then $z \in I$, a contradiction. Thus $x_i \in P$, but $i = r-1$ or $i = r$ so $(x_1, \ldots, x_{r-1}) \subseteq I$ contradicts $\operatorname{codim}(I) = r - 2$. Hence f_i must have a second term for $i = r, r-1$.

Form the S-polynomial for f_{r-1} and f_r, that is

$$x_r f_{r-1} - x_{r-1} f_r = x_r h_{r-1} - x_{r-1} h_r \in P. \tag{1}$$

Since the S-polynomial is in P its initial term is in I.

Suppose all of the monomials that form $x_r h_{r-1}$ and those that form $x_{r-1} h_r$ are the same. Then x_r divides h_r and x_{r-1} divides h_{r-1}. Therefore $x_{r-1}(z + h_{r-1}/x_{r-1}) \in P$ and hence $x_{r-1} \in P$ or $z + h_{r-1}/x_{r-1} \in P$. The first case implies $(x_1, \ldots, x_{r-1}) \in I$ and the second case implies $z \in I$ which are both contradictions. Write $h_{r-1} = r_1 m_1 + \ldots + r_s m_s$ where $r_1, \ldots r_s \in k$ and $m_1, \ldots, m_s$ are monomials such that $m_1 > \ldots > m_s$ and similarly write $h_r = s_1 n_1 + \ldots + s_t n_t$, $s_1, \ldots, s_t \in k$, $n_1 > \ldots > n_t$ monomials. Choose i smallest such that either $x_r m_i = x_{r-1} n_i$ and $r_i \neq s_i$ or $x_r m_i \neq x_{r-1} n_i$. We consider two cases.

(1): $x_r m_i = x_{r-1} n_i$ and $r_i \neq s_i$ or $x_r m_i > x_{r-1} n_i$ so $x_r m_i$ is the leading monomial of the S-polynomial $S(f_{r-1}, f_r)$.

(2): $x_{r-1} n_i > x_r m_i$ so $x_{r-1} n_i$ is the leading monomial of $S(f_{r-1}, f_r)$.

In case (1) $x_r m_i$ is in I. This implies $x_j m_i \in I$ for $1 \leq j \leq r$ since I is Borel-fixed (Proposition 3). Hence $m_i(x_1, \ldots, x_r) \subseteq I$ which implies $(\overline{x_1}, \ldots, \overline{x_r}) = \operatorname{ann}_{R/I}(\overline{m_i})$ and hence $\overline{m_i}$ is in the socle of R/I. The total degree of $x_{r-1} z$ is the same as that for m_i since f_{r-1} is homogeneous, so the total degree of m_i is strictly larger than the total degree of z. This contradicts our choice of z in the socle.

In case (2) $x_{r-1}n_i$ is in I. Again, $x_i n_i \in I$ for $1 \leq i \leq r-1$ and hence $(\overline{x_1}, \ldots, \overline{x_{r-1}}) \subseteq \mathrm{ann}_{R/I}(\overline{n_i})$. Since $n_i \notin I$, if we have equality then $(x_1, \ldots, x_{r-1}) \in Ass(R/I)$. If the containment $(\overline{x_1}, \ldots, \overline{x_{r-1}}) \subset \mathrm{ann}_{R/I}(\overline{n_i})(\overline{x_1}, \ldots, \overline{x_{r-1}})$ is strict then $\overline{x_r}^t$ is in $\mathrm{ann}_{R/I}(\overline{n_i})$ for some $t \geq 1$. Choose t least such that $\overline{x_r}^t \in \mathrm{ann}_{R/I}(\overline{n_i})$. Then $x_r^t n_i \in I$ and hence $x_j x_r^{t-1} n_i \in (I)$ for $1 \leq j \leq r$. This implies

$$(\overline{x_1}, \ldots, \overline{x_r}) = \mathrm{ann}_{R/I}(\overline{x_r^{t-1} n_i}), \tag{2}$$

and since $x_r^{t-1} n_i \notin I$ by the minimality of t, $x_r^{t-1} n_i$ is in the socle of R/I. The total degree of $x_r^{t-1} n_i$ is greater than or equal to that of n_i and hence strictly greater than the total degree of z. Therefore, Equation 2 gives a contradiction to our choice of z and $\mathrm{ann}_{R/I}(\overline{n_i}) = (\overline{x_1}, \ldots, \overline{x_{r-1}})$ as desired. □

References

[BS] D. Bayer and M. Stillman, A criterion for detecting m-regularity, *Invent. Math.* **87**(1987), 1-11.

[E] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150. Springer-Verlag, New York, 1996.

[Gre] M. L. Green, Generic Initial Ideals, *Six Lectures on Commutative Algebra*, Birkhäuser Verlag, 1998.

[HT] S. Hosten and R. Thomas, The associated primes of initial ideals of lattice ideals, *Math. Res. Let.* **6**(1999), no. 1, 83-97.

[KS] M. Kalkbrenner and B. Sturmfels, Initial complexes of prime ideals, *Adv. in Math.* **116**(1995), no. 2, 365-376.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
E-mail address: ataylor@math.rutgers.edu